On polynomially integrable billiards on surfaces of constant curvature

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The famous Birkhoff Conjecture deals with convex bounded planar billiard Ω with smooth boundary. A particle moves in Ω with constant speed, and as it hits the boundary, it reflects and moves in the reflected direction with velocity of the same module etc. A billiard is *Birkhoff integrable,* if the above dynamical system has a first integral independent with the module of the speed on a neighborhood of the unit tangent bundle to the boundary. If $\partial\Omega$ is an ellipse, then there exists a non-trivial integral quadratic in the velocity. The Birkhoff Conjecture states that *every Birkhoff integrable planar billiard is an ellipse.* Recently V.Kaloshin and A.Sorrentino proved its local version: every Birkhoff integrable deformation of an ellipse is an ellipse [5].

The polynomial version of the Birkhoff Conjecture, which was first stated and studied by Sergey Bolotin in 1990, concerns *polynomially integrable* billiards, where there exists a first integral polynomial in the velocity that is non-constant on the unit level hypersurface of the module of the velocity.

In this talk we present a brief survey of Birkhoff Conjecture and a complete solution of its polynomial version. We prove that *each bounded polynomially integrable planar billiard with C^2-smooth non-linear connected boundary is an ellipse.* Analogous result is obtained for billiards on surfaces of constant curvature (plane, sphere, hyperbolic plane), with complete classification of polynomially integrable billiards with piecewise smooth boundary. These are joint results with Mikhail Bialy and Andrey Mironov [1, 2, 3, 4].

References

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